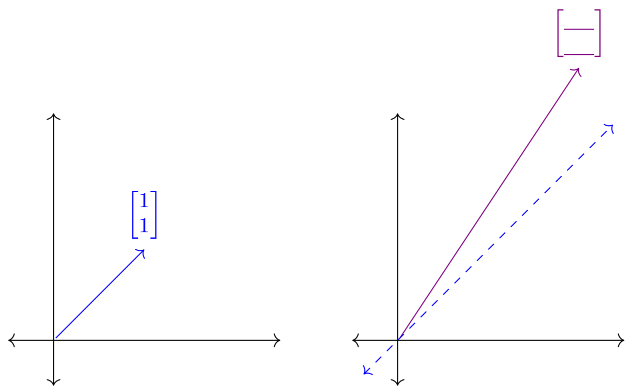
Math 4B Week 7 Guided Practice
Linear Algebra Review - Eigenvectors and Their Eigenvalues

The guided practice problems below were taken from Math 4A.

Examples, and visualizing what it means to be an eigenvector.

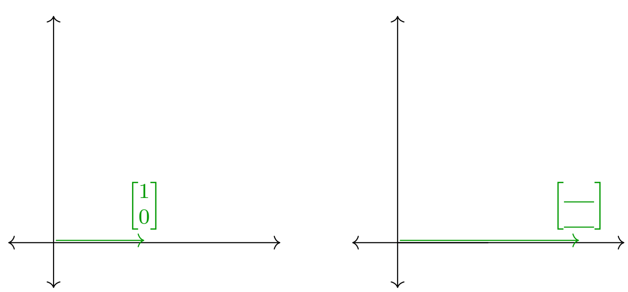
1. Consider the matrix $\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$. In order for a vector to be an eigenvector for this matrix, it needs to come out of the matrix (or linear transformation) parallel to its self. For example, $\begin{bmatrix}1\\1\end{bmatrix}$ is **not** an eigenvector:

$$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \qquad = \begin{bmatrix}\underline{\quad}\\\underline{\quad}\end{bmatrix}$$



On the other hand, $\begin{bmatrix}1\\0\end{bmatrix}$ is:

$$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \qquad = \begin{bmatrix}\underline{\quad}\\\underline{\quad}\end{bmatrix}$$



In fact, $\begin{bmatrix}c_1\\0\end{bmatrix} = c_1 \cdot \begin{bmatrix}\underline{\quad}\\\underline{\quad}\end{bmatrix}$, so for any scalar c_1,

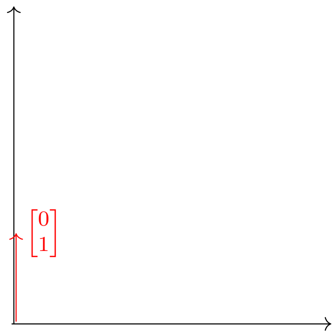
$$\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}c_1\\0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\left(c_1 \cdot \begin{bmatrix}1\\0\end{bmatrix}\right) = c_1 \cdot \left(\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right) = c_1 \cdot \begin{bmatrix}\underline{\quad}\\\underline{\quad}\end{bmatrix} = 2 \cdot \begin{bmatrix}\underline{\quad}\\\underline{\quad}\end{bmatrix}.$$

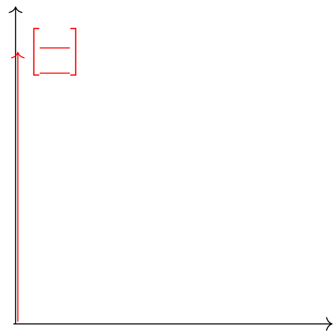
This means the entire x-axis, or $c_1 \cdot \begin{bmatrix}1\\0\end{bmatrix}$ varied over every c_1, gives us solutions to the equation

$$A\mathbf{x} = 2 \cdot \mathbf{x}.$$

How about an eigenvector that isn't on the x-axis?
With a diagonal matrix like this one, eigenvectors are easy to find:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad\qquad = \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \end{bmatrix}$$





How about $c_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for any c_2?

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \left(c_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = c_c \cdot \left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = c_2 \cdot \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \end{bmatrix} = 3 \cdot \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \end{bmatrix}.$$

It's worth noting that both of these lines of eigenvectors, each for a different eigenvalue (2 and 3 respectively) are subspaces. In fact...

2. (a) Find the null-space for the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) Find the null-space for the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

   (c) How do these compare to the solution sets for the equations below?

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

There's an important connection here that's worth articulating. What is the relationship between the lines of eigenvectors above and the two nullspaces you found? Why do you think that is?

3. (a) Can you find all the eigenvectors for the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$? No computational tricks necessary.

   (b) How about the eigenvectors for the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$?

   (c) Both of these solution sets are the null space of some matrix, and in fact it's the same matrix for both! What matrix must that be?

4. (a) Now, can you find two different eigenvectors of the matrix $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ with **different** eigenvalues?

   (b) How about for the slightly more difficult case with this matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$?

How Do We Find Them?

This last question was a challenge problem from the beginning of the quarter. And even in week two, many of you could solve this problem if you already knew the eigenvalues.

How to find eigenvectors when you already know the eigenvalue:
Let's find all the eigenvectors with eigenvalue 3: Non-trivial solutions to

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{bmatrix} = \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix}$$

$$\begin{bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And you can solve from there, or write it as the matrix equation:

$$\begin{bmatrix} \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and then solve as an augmented matrix problem.

More generally, for any matrix A, if you know that λ is an eigenvalue of A then you can find all of the eigenvectors of A with eigenvalue λ by solving the equation

$$A\mathbf{x} = \lambda \cdot \mathbf{x}.$$

In solving this, you may notice that you are moving the variables in the right side of the equation over to the left side and in doing so, the x_1 coefficient subtracts from the x_1 coefficient on the first row, the x_2 coefficient subtracts from the x_2 coefficient on the second row, etc. It will save you time in the long run to set up the equation a little differently by first rewriting $\lambda \cdot \mathbf{x}$ as $(\lambda \cdot I)\mathbf{x}$, where $\lambda \cdot I$ is the scaled identity matrix (with λ's on the diagonal), and then subtract from both sides.

$$A\mathbf{x} - (\lambda \cdot I)\mathbf{x} = (A - \lambda \cdot I)\mathbf{x} = \mathbf{0}$$

And here, you are just finding the kernel of the slightly different matrix $A - \lambda \cdot I$, which is A with λ's taken away from its diagonal entries.
And now the problem is almost exactly like the problems you have done many times before! It's just another kernel problem.

This works nicely, but it assumes we already know the eigenvalues. If we had a reliable way to find these we would be set. So how do we do this?
How to find eigenvalues:
How would we have found eigenvalues for the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ if we did not already have an idea of where to look? We would be searching for a scalar λ that would make the equation

$$A\mathbf{x} = \lambda \cdot \mathbf{x}$$

have a solution $\mathbf{x}$ other than just the trivial solution ($\mathbf{0}$ is always a solution but doesn't count as an eigenvector). Trying to solve this by brute force or trial and error can be very difficult in general. However, the solution has a nice analog to something you learned in algebra. A problem like

$$x^2 - 5x = -6.$$

You didn't just want to plug in values for x to see if you ever got -6. The key to cracking this type of problem in algebra was the initial finess of posing this difficult question as an easier question:

$$x^2 - 5x + 6 = 0 \longrightarrow x^2 - 2x - 3x + 6 = 0 \longrightarrow x(x-2) - 3(x-2) = 0$$

$$\longrightarrow (x-3)(x-2) = 0.$$

$(x-3)(x-2) = 0$ when either $(x-3) = 0$ or $(x-2) = 0$, so either $x = 2$ or $x = 3$, giving us the two solutions.

The idea behind finding the answer was finding what gave you zero. And with the eigenvalue problem, this strategy also works wonders. Let's turn this problem into a "*stuff* $= 0$" problem:

$$A\mathbf{x} = \lambda \cdot \mathbf{x} \longrightarrow A\mathbf{x} - \lambda \cdot \mathbf{x} = \mathbf{0} \longrightarrow A\mathbf{x} - (\lambda \cdot I)\mathbf{x} = \mathbf{0}$$

$$\longrightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}.$$

And now we only need to find when $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has non-trivial solutions. This is exactly when there are non-zero vectors in the kernel, i.e. when $A - \lambda I$ is not one-to-one. And for square matrices, this lends its self exceptionally well to the determinant test, which is what this page was all about getting to.

Let the determinant determine it

λ is an eigenvalue for A exactly when $|A - \lambda I| = 0$.
For the above matrix this means

$$\left| \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 1 \cdot 0 = (\lambda - 2)(\lambda - 3) = 0,$$

which has solutions $\lambda = 2$ and $\lambda = 3$. (Yes, all upper-triangular matrices have calculations this easy. What do you think for lower triangular ones?)

1. Find all eigenvalues and eigenvectors for the following matrices:

   (a) $\begin{bmatrix} 4 & 2 \\ 0 & -7 \end{bmatrix}$

   (b) $\begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Coordinates with Eigenvectors

You may have noticed that it is much easier to find eigenvalues and eigenvectors for upper or lower triangular matrices, with the best case being diagonal matrices. Diagonal matrices are especially nice because they send each basis vector $\mathbf{e_i}$ to $d_i \cdot \mathbf{e_i}$. And if you remember, the e_i's make up exactly the basis for $\mathbb{R}^n$ that gives us a coordinate system that matches the standard coordinate system for $\mathbb{R}^n$ (the entries in each column vector). But we don't need each of our e_i's to be eigenvectors for things to be this nice. If we can find any basis of eigenvectors for our vector space, then the linear operator we're looking at has a really nice matrix form that imitates a diagonal matrix really well.

Imagine you have a basis of eigenvectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and you want to see what happens to any vector you put through your linear transformation. Any vector $\mathbf{x}$ I want to put into my linear transformation I can write as a unique linear combination

$$\mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$$

and I can put that through the linear transformation T:

$$T(\mathbf{x}) = T(c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}) = c_1 \cdot T(\mathbf{v_1}) + c_2 \cdot T(\mathbf{v_2}) + c_3 \cdot T(\mathbf{v_3})$$

$$= c_1(\lambda_1\mathbf{v_1}) + c_2(\lambda_2\mathbf{v_2}) + c_3(\lambda_3\mathbf{v_3})$$

and this gives you the coordinates of $T(\mathbf{x})$:

$$(c_1\lambda_1)v_1 + (c_2\lambda_2)v_2 + (c_3\lambda_3)v_3.$$

There is only one matrix that sends $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ to $\begin{bmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \lambda_3 c_3 \end{bmatrix}$, and it's

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Condensed Recap and Some Key Terms

1. **Eigenvectors** for a matrix A are vectors with the special property that when they get put into the linear transformation they are moved to some vector in the same direction.

2. Explicitly, $\mathbf{v}$ is an eigenvector of the linear transformation with matrix A if $A\mathbf{v} = \lambda \cdot \mathbf{v}$.

3. λ is called $\mathbf{v}$'s **eigenvalue**.

4. A very helpful tool for finding eigenvalues and eigenvalues is a **difference matrix** $A - \lambda I$, which sends a vector $\mathbf{v}$ the the difference between where A sends $\mathbf{v}$ and what scaling by λ does to $\mathbf{v}$. In particular,

   $$\boxed{\mathbf{v} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda \text{ if and only if } \mathbf{v} \text{ is in the kernel of } A - \lambda I.}$$

5. Once you know an eigenvalue λ for a matrix A, you can find all of the eigenvectors by just finding the kernel of the difference matrix $A - \lambda I$. But it is hard to find eigenvectors without first knowing what A's eigenvalues are.

6. To find A's eigenvalues, take the determinant of the difference matrix (called the **characteristic polynomial**) and set it equal to zero we have the the equation

   $$|A - \lambda I| = 0.$$

   The book calls this the **characteristic equation**.

7. If a linear factor shows up twice in the characteristic polynomial, i.e. for example, you see the factor $(\lambda - 2)^2$ and $\lambda - 2$ does not appear as another factor, then your **eigenspace** for $\lambda = 2$ could be as large as two dimensions. How would you know? Find a basis for the kernel of $A - 2I$ and that's the dimension!

8. The number of times a linear factor, like $(\lambda - 2)$, shows up in your characteristic polynomial is called its **algebraic multiplicity** and it suggests an upper limit for how big the eigenspace for $\lambda = 2$ could be. The actual dimension of your eigenspace could be lower (which is really annoying), and we call that dimension the **geometric multiplicity** for the eigenvalue $\lambda = 2$.

9. It seems like there should never be a difference between the algebraic multiplicity and the geometric multiplicity, but there often is. When the are different it means that we don't have enough linearly independent eigenvectors to form a basis. This makes us sad pandas. All of the cool stuff we will be doing with diagonalizing won't apply to these matrices. If you're wondering how a disparity between these multiplicities can exist in the first place, we're unfortunately at a loss to describe these "missing eigenvectors" in terms of the original definition of an eigenvector. But if you change your definition to vectors in the kernel of the difference matrix (this is how we find our eigenvectors), then it turns out there is a really easy way to generalize the notion of eigenvector and these missing vectors appear! Unfortunately, we need to wait for a later class to learn about these hidden generalizations of eigenvectors.

Diagonalizing Examples

1. Motivation: Find the 5th power of the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

   Explanation: When in doubt, ask yourself where A^5 sends e_1 and e_2. We'll start with e_1: A doubles e_1, so A sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is then sent by A to $2 \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Aha, a pattern! e_1 is sent to $\begin{bmatrix} 2^5 \\ 0 \end{bmatrix}$.

   For similar reasons, e_2 is sent to $(-1)^5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. So...

   Answer: $A^5 = \begin{bmatrix} 2^5 & 0 \\ 0 & (-1)^5 \end{bmatrix}$.

2. Diagonalize the matrix $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then find a formula for the entries of A^n.

   Explanation: We need to find the eigenvalues and eigenvectors first.

   Charecteristic Equation
   We take the determinant of the difference matrix. You should practice computing this and factoring the equation now instead of just reading the results.

   $$|A - \lambda I| = (\lambda - 3)(\lambda + 1) = 0$$

   $\lambda = 3$

   (a) Compute the **difference matrix**.

   $$A - 3I = \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

   (b) Find its **kernel**

   Row Reduction:

   $$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}\cdot \text{R1}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{2\text{R1}+\text{R2}\to\text{R2}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

   So the kernel is $\left\{ s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. This also gives us an eigenvector to pair with 3. It's $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$*.

   $\lambda = -1$

   (a) Compute the **difference matrix**.

   $$A - (-1)I = \begin{bmatrix} 1+1 & 2 \\ 2 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

   (b) Find its **kernel**

   $$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot \text{R1}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{-2\text{R1}+\text{R2}\to\text{R2}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

   So the kernel is $\left\{ s \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, which has a basis of $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

*There are many other eigenvectors that would also work. In fact, any non-zero scalar multiple of this choice is fine.

Diagonalizing the Matrix
Now we're ready to diagonalize! The idea behind diagonalization is that if we use the basis of eigenvectors $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ then our linear transformation would be very simple to understand. It just scales the first basis vector by 3 and the second by -1. So with this basis, it would have the matrix

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

The trick is that we have to figure out the change-of-coordinate matrices to go from coordinates in the standard basis to coordinates in β. We also have to find the matrix that changes the coordinates back. Fortunately, this second matrix is easy to find: It's just the matrix of eigenvectors!

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

This also makes it easy to find P^{-1} because we can just take the inverse of P. We'll find the determinant and use that with the 2×2 inverse formula.

$$|P| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

We've got it! Now we can write A as a diagonalized matrix:

$$A = PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right).$$

Computing Powers of A
Now that we know that A has the diagonal form above, we can compute high powers of A as follows:

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}^n \left(\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right) \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 3^n \\ (-1)^{n+1} & (-1)^n \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3^n + (-1)^{n+2} & 3^n + (-1)^{n+1} \\ 3^n + (-1)^{n+1} & 3^n + (-1)^n \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3^n + (-1)^n & 3^n - (-1)^n \\ 3^n - (-1)^n & 3^n + (-1)^n \end{bmatrix}. \end{aligned}$$

This gives us a neat formula, with the $\frac{1}{2}$ factored out of the matrix. However, we could distribute it to each of the terms to be more explicit.

Answer: $A^n = \begin{bmatrix} \frac{3^n+(-1)^n}{2} & \frac{3^n-(-1)^n}{2} \\ \frac{3^n-(-1)^n}{2} & \frac{3^n+(-1)^n}{2} \end{bmatrix}.$

3. Your turn! Diagonalize and find a formula for the nth power of these matrices: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 0 & -7 \end{bmatrix}.$

Fibonacci Example We would like to find an efficient way to calculate large Fibonacci numbers. In particular, we would like to know what the 1000th Fibonacci number is and see how close it is to the golden ratio times the 999th. The default method is to just compute each term by hand. This is not a very fun way because you have to compute every single Fibonacci number in between. Let's use linear algebra!

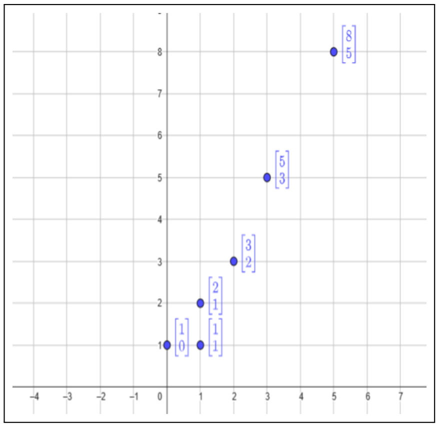


Figure 1: Fibonacci Pairs

Given any pair of consecutive Fibonacci numbers $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$, the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ will give us $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$. This works to find later Fibonacci powers because you only need two consecutive Fibonacci numbers to find the later Fibonacci numbers. A strategy with just one term will not work because you wouldn't have enough information to compute the subsequent terms. Here's how the matrix works.
For $(1,1)$ to $(1,2)$, we can compute:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

From $(5,8)$ to $(8,13)$ we can compute:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 8 + 1 \cdot 5 \\ 1 \cdot 8 + 0 \cdot 5 \end{bmatrix} = \begin{bmatrix} 8+5 \\ 8 \end{bmatrix} = \begin{bmatrix} 13 \\ 8 \end{bmatrix}.$$

And for a general consecutive pair F_{n-1} and F_n, we have:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n + 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

As you can see, the 1's on the top row of the matrix give us a top result that comes from adding one of each of the previous two entries. That's exactly how we find the next Fibonacci number. And on the bottom row, we have 1 of the most recent Fibonacci number and 0 of the other. This moves the newest Fibonacci number to the bottom slot, which is also exactly what we want.
We can A plot of adjacent of Fibonocci numbers as points, which you can see in the picture below. I switched the order of the coordinates because I wanted the golden ratio axis to be a line through the origin with slope being the golden ratio.
If we take the determinant of the difference matrix (we call this the **characteristic polynomial**) and set it equal to zero we have the the characteristic equation

$$\lambda^2 - \lambda - 1 = 0.$$

Interesting...the equation above is the same as the equation used to find the **golden ratio**. In fact, the two solutions are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. These are usually written as φ and $-\varphi^{-1}$ respectively, where φ is the golden ratio. The second solution, $-\varphi^{-1} = \frac{1-\sqrt{5}}{2}$ is actually another solution to the golden ratio equation, but it is usually thrown out because it's negative. We will definitely make use of it here because it's one of our two eigenvalues!

$\lambda = \varphi$

(a) Compute the **difference matrix**.

$$A - \varphi I = \begin{bmatrix} 1 - \left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & 0 - \left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}$$

(b) Find its **kernel**
Now we find a basis for the kernel of this difference matrix. After struggling with square roots of 5 and fractions with my own scratch work last week and over the weekend, I came to the conclusion that it's a lot easier to use φ instead of $\frac{1+\sqrt{5}}{2}$, and $-\varphi^{-1}$ instead of $\frac{1-\sqrt{5}}{2}$, so I'll be using those. You will usually have easier numbers to work with, so don't worry so much about this part.
Row Reduction:

$$\left[\begin{array}{cc|c} \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & -\frac{1+\sqrt{5}}{2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} -\varphi^{-1} & 1 & 0 \\ 1 & -\varphi & 0 \end{array}\right] \xrightarrow{-\varphi\cdot \text{R1}} \left[\begin{array}{cc|c} 1 & -\varphi & 0 \\ 1 & -\varphi & 0 \end{array}\right] \xrightarrow{\text{-R1+R2}\to\text{R2}} \left[\begin{array}{cc|c} 1 & -\varphi & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So the kernel is $\left\{ s\begin{bmatrix}\varphi\\1\end{bmatrix}\right\}$. This also gives us an eigenvector to pair with φ. It's $\begin{bmatrix}\varphi\\1\end{bmatrix}$*.

Next we'll find the eigenspace for $\lambda = -\varphi^{-1}$, and once we've done that we'll have a basis of eigenvectors.

$\lambda = -\varphi^{-1}$

(a) Compute the **difference matrix**.

$$A - \varphi I = \begin{bmatrix} 1-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right) & 1 \\ 1 & 0-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 \\ 1 & -\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\end{bmatrix} = \begin{bmatrix}\varphi & 1 \\ 1 & -(-\varphi^{-1})\end{bmatrix} = \begin{bmatrix}\varphi & 1\\ 1 & \varphi^{-1}\end{bmatrix}$$

(b) Find its **kernel**

$$\left[\begin{array}{cc|c} \varphi & 1 & 0 \\ 1 & \varphi^{-1} & 0 \end{array}\right] \xrightarrow{\varphi^{-1}\cdot\text{R1}} \left[\begin{array}{cc|c} 1 & \varphi^{-1} & 0 \\ 1 & \varphi^{-1} & 0 \end{array}\right] \xrightarrow{\text{-R1+R2}\to\text{R2}} \left[\begin{array}{cc|c} 1 & \varphi^{-1} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So the kernel is $\left\{ s\begin{bmatrix}-\varphi\\1\end{bmatrix}\right\}$, which has a basis of $\left\{ \begin{bmatrix}-\varphi\\1\end{bmatrix}\right\}$.

Diagonalizing

Now we're ready to diagonalize! The idea behind diagonalization is that if we use the basis of eigenvectors $\beta = \left\{\begin{bmatrix}\varphi\\1\end{bmatrix}, \begin{bmatrix}-\varphi\\1\end{bmatrix}\right\}$ then our linear transformation would be very simple to understand. It just scales the first basis vector by φ and the second by $-\varphi^{-1}$. In the picture to the left, the vectors on the golden axes are scaled by φ and the vectors on the purple axis are scaled by $-\varphi^{-1}$. This is why we want this basis! From the perspective of this coordinate system, our matrix would just be $D = \begin{bmatrix}\varphi & 0\\ 0 & -\varphi^{-1}\end{bmatrix}$.

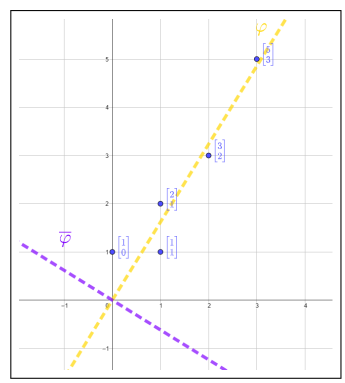


Figure 2: Growth and Decay

In order to use this simpler diagonal matrix instead, we just need to figure out the change-of-coordinate matrices to go from the standard basis into β coordinates. Okay, we'll also need the matrix that changes the coordinates back. Fortunately, the second matrix is easy to find. It's just the matrix of eigenvectors

$$P = \begin{bmatrix}\varphi & -\varphi\\ 1 & 1\end{bmatrix}.$$

In case you've forgotten the regular way to find it, this also makes it easy to find P^{-1} because we can just take the inverse of P. We'll find the determinant and use that with the 2×2 inverse formula.

$$|P| = \begin{vmatrix}\varphi & -\varphi\\ 1 & 1\end{vmatrix} = 2\varphi$$

$$P^{-1} = \frac{1}{|P|}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix} = \frac{1}{2\varphi}\begin{bmatrix}1 & \varphi\\ -1 & \varphi\end{bmatrix}.$$

Now we can write A as a diagonalized matrix:

$$A = PDP^{-1} = \begin{bmatrix}\varphi & -\varphi\\ 1 & 1\end{bmatrix}\begin{bmatrix}\varphi & 0\\ 0 & -\varphi^{-1}\end{bmatrix}\left(\frac{1}{2\varphi}\begin{bmatrix}1 & \varphi\\ -1 & \varphi\end{bmatrix}\right).$$

*There are many other eigenvectors that would also work. In fact, any non-zero vector on the same axis would be fine.

If we're starting with the pair of Fibonacci numbers $\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we would need to apply the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ exactly 999 times to get $\begin{bmatrix} F_{1000} \\ F_{999} \end{bmatrix}$. But we don't need 999 matrices, we only need three:

$$A^{999} = PD^{999}P^{-1} = \frac{1}{2\varphi} \begin{bmatrix} \varphi & -\varphi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^{999} & 0 \\ 0 & (-\varphi)^{-999} \end{bmatrix} \begin{bmatrix} 1 & \varphi \\ -1 & \varphi \end{bmatrix}.$$

Note that this works because diagonal matrices just scale their axes over and over again. We could **not** just do this with the original matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{999} \neq \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1^{999} & 1^{999} \\ 1^{999} & 0^{999} \end{bmatrix}.$$

Applying this to the first two Fibonacci numbers, we have:

$$\begin{aligned} &\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \\ &= A^k \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\ &= \frac{1}{2\varphi} \begin{bmatrix} \varphi & -\varphi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^k & 0 \\ 0 & (-\varphi)^{-k} \end{bmatrix} \begin{bmatrix} 1 & \varphi \\ -1 & \varphi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2\varphi} \begin{bmatrix} \varphi & -\varphi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^k & 0 \\ 0 & (-\varphi)^{-k} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{2\varphi} \begin{bmatrix} \varphi & -\varphi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^k \\ -(-\varphi)^{-k} \end{bmatrix} \\ &= \frac{1}{2\varphi} \begin{bmatrix} \varphi\varphi^k + \varphi(-\varphi)^{-k} \\ \varphi^k - (-\varphi)^{-k} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\varphi^k + (-\varphi)^{-k}}{2} \\ \frac{\varphi^k - (-\varphi)^{-k}}{2\varphi} \end{bmatrix} \end{aligned}$$

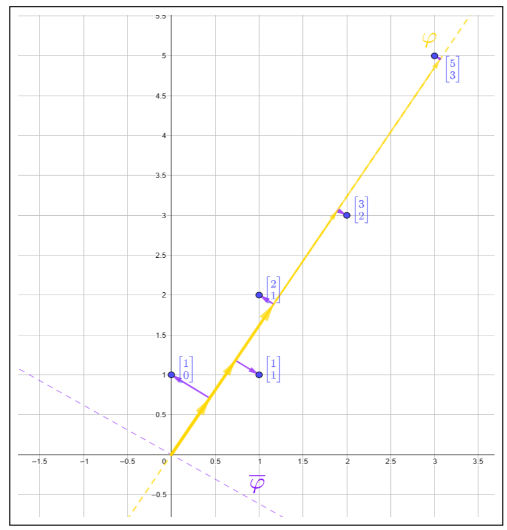


Figure 3: Eigen Axes

Setting k equal to 999, we get the following:

$$\boxed{\mathrm{F}_{1000} = \frac{\varphi^{1000} + (-\varphi)^{-100}}{2}} \text{ and } \boxed{\mathrm{F}_{999} = \frac{\varphi^{1000} - (-\varphi)^{-1000}}{2\varphi}}.$$

In fact, the two column entries give us two different closed formulas for F_n, the top one when you make $k = (n-1)$ and the bottom one when you make $k = (n-2)$.

The Golden Ratio

Given the fact that $-\varphi^{-1} \approx -.618$, its powers rapidly decay: $(-\varphi)^{-10} \approx 0.008$ and $(-\varphi)^{-1000} \approx 0.000005$. So we can confidently approximate $F_{1000} \approx \frac{\varphi^{1000}}{2}$ and $F_{999} \approx \frac{\varphi^{1000}}{2\varphi}$, which means $\frac{F_{1000}}{F_{999}} \approx \varphi$. This approximation is very good. In fact, these pairs move toward the golden axis you can see in Figure 3 very quickly. Each time the matrix is applied, the part of the Fibonacci pair that is made up of the golden vector grows by about 40% while the part made up of the purple vector shrinks by about 40%. This quickly compounds, and the purple part is barely visible even for the pair $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Condensed Recap and Some Key Terms

1. A square matrix is **diagonalizable** if the domain(/codomain) has a basis of eigenvectors. This amounts to being the same thing as the dimension of each eigenspace matching the number of times that eigenvalue appears as a linear factor in the characteristic polynomial.

2. Two matrices are said to be **similar** if they do the same thing relative to a change of coordinates. Specifically, a matrix A is similar to a matrix B if $A = PBP^{-1}$ for some change-of-coordinates matrix P.

3. Using the terminology of similarity, a square matrix is diagonalizable if and only if it is similar to a diagonal matrix. Specifically, $A = PDP^{-1}$ for some diagonal matrix D.

A Preview of Math 4B

This whole time, we've been discussing eigenvectors of a matrix. But the linear transformation does not need to come from a matrix. It could be any linear transformation, as long as the domain and codomain are the same (which is the condition we had for $\mathbb{R}^m = \mathbb{R}^n$ that makes our matrices square matrices).
The derivative is an important linear transformation, and understanding its eigenvalues and eigenvectors is incredibly helpful for Math 4B.

1. Find an eigenvector with eigenvalue $\lambda = 2$ for the linear transformation $T(f) = f'$.

   Answer: $f(x) = e^{2x}$ is an eigenvector, since $\frac{d}{dx}(e^{2x}) = 2 \cdot e^{2x}$.

More generally, for any λ we have $\frac{d}{dx}(e^{\lambda x}) = \lambda \cdot e^{\lambda x}$, so these are all eigenvectors of the derivative. Knowing this is very useful for solving the differential equation

$$y' = \lambda y.$$

If you remember finding the kernel of the difference matrix, that comes up again too: These functions are also solutions to the linear differential equations

$$T(y) = y' - \lambda y = 0.$$

These equations are homogeneous (because you're looking for trivial solutions), but you can also use their homogeneous solutions for non-homogeneous equations by adding them to any single non-homogeneous solution you might find. For example, if you know that $y = x$ is a solution to the equation $y' - 2y = -2x + 1$, then you can add the homogeneous solutions $y = s(e^{2x})$ to your solution and get the full solution set $y = s(e^{2x}) + x$.
If you thought that was useful, it gets much better. You will see higher order versions of this linear differential equation midway through math 4B, and these "eigenfunctions" will be the basis (pun intended) for your solution sets. You'll factor polynomials a lot like the characteristic polynomial to find these eigenfunctions, and at the end of the quarter you will be solving first order systems of linear differential equations that look like

$$x' = 6x + 3y$$
$$y' = -x + 2y$$

and can be written as:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 6 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Ironically, though these kinds of problems appear towards the end of the quarter, the skills you learn now are all you need to find eigenvalues 5 and 3 for the matrix, and their respective eigenvectors of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. If you're curious, with this information you can get a basis for the entire solution set. Just multiply the eigenfunctions and eigenvectors. It looks like

$$\textit{Solution Basis} = \left(e^{5t} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \quad e^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

with a full solution set of

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{5t} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} + C_2 e^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Unfortunately, even though you will learn almost everything you need to know to solve these problems in the next week or two, most Math 4B students forget their linear algebra half way through the quarter and have to learn all of this 10 weeks later.
If all of this made sense, wonderful! If not, no worries. Just knowing that the functions $e^{\lambda x}$ as eigenvectors for the derivative will help you a lot when you take Math 4B.